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New Classes of Globally Convexized Filled Functions for Global Optimization

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Abstract. We propose new classes of globally convexized filled functions. Unlike the globally convexized filled functions previously proposed in literature, the ones proposed in this paper are continuously differentiable and, under suitable assumptions, their unconstrained minimization allows to escape from any local minima of the original objective function. Moreover we show that the properties of the proposed functions can be extended to the case of box constrained minimization problems. We also report the results of a preliminary numerical experience.

Key words: Filled functions; Global optimization; Nonlinear optimization

1. Introduction

Several real world applications need the solution of global optimization problems. However the definition of an efficient method for such problems is still an open question. Many different approaches have been proposed in literature to solve this class of difficult problems. One of these is based on the use of the *filled functions*. These methods have been initially introduced in Ge (1990), Ge and Qin (1987, 1990), and recently reconsidered in Liu (2001).

The idea behind the filled functions is to construct an auxiliary function that allows us to escape from a given local minimum x_1^* of the original objective function $f(x)$.

In this work we try to extend the approach proposed in Ge and Qin (1990), since, in our opinion, the particular filled functions there introduced show interesting theoretical properties. This class of filled functions $U(x, x_1^*, \tau, \rho)$ depends on the local minimum x_1^* of $f(x)$ and on two parameters, $\tau, \rho > 0$. If parameter ρ is chosen properly and x_1^* is not a global minimum of the objective function $f(x)$, then $U(x, x_1^*, \tau, \rho)$ has global minimum points $\bar{x}$ where $f(\bar{x}) < f(x_1^*)$. Moreover if parameter τ is greater than a threshold value, which depends on the behaviour of $f(x)$ on a compact set Ω, then $U(x, x_1^*, \tau, \rho)$ has no unconstrained stationary points $\hat{x} \in \Omega$ where $f(\hat{x}) \geqslant f(x_1^*)$ except a prefixed point x_0. However the approach proposed in Ge and Qin (1990) has some drawbacks:

- the introduced filled functions are not smooth and therefore they are not easy to minimize by using standard code;

- the authors use, as compact set Ω, the level set of the original objective function $f(x)$ and this implies that a sequence of points produced by an unconstrained algorithm may be attracted towards a stationary point of $U(x, x_1^*, A, h)$ out of Ω. This difficulty cannot be solved by a constrained minimization, because the constrained stationary points of $U(x, x_1^*, A, h)$ have no connection with local minimum points of $f(x)$.

Our first aim has been to define new classes of filled functions that overcome these drawbacks, and then to extend our analysis to the case of box constrained minimization problems.

More in particular, in Section 2 we define two new classes of filled functions for unconstrained global optimization problems; in Section 3 we adapt these functions for solving box constrained minimization problems. Finally in Section 4 we report the result of a preliminary numerical experience showing the practicability of the proposed approach.

In the sequel we will denote by $\|x\|$ the standard Euclidean norm of x. In Ge and Qin (1990) a function is called globally convex if it satisfies the following definition:

DEFINITION. A function $f: R^n \to R$ is globally convex if

$$\lim_{\|x\| \to +\infty} f(x) = +\infty .$$

In literature a globally convex function is usually named a coercive function.

2. A class of continuously differentiable filled function for unconstrained minimization problems

In this section we consider the following unconstrained minimization problem

$$\begin{aligned} &\min \ f(x) \\ &x \in R^n \end{aligned} \tag{1}$$

where $f: R^n \to R$ is a smooth globally convex function.

Drawing inspiration from Ge and Qin (1990), we introduce two classes of filled functions, whose expressions are obtained by combining two elementary functions, $\eta(t)$ and $\varphi(t)$. In order to guarantee theoretical properties of our classes of filled functions, $\eta(t)$ needs to satisfy the following properties:

(1) $\eta(0) = 0$
(2) $\eta'(t) \geq \tilde{a} > 0$ per $t > 0$

Some examples of function $\eta(t)$ satisfying properties (1) and (2) are the following:

- $\eta(t) = t$
- $\eta(t) = \tan t$
- $\eta(t) = e^t - 1$

As regards the assumptions on $\varphi(t)$, they depend by the structure of the filled function. In particular $\varphi(t)$ needs to satisfy some of the following properties:

(1) $\varphi'(t) \geqslant 0 \; \forall t \in [0, +\infty)$ and $\varphi'(t) > 0 \; \forall t \in [-t_1, 0)$ with $t_1 > 0$;
(2) $\lim_{t\to+\infty} \varphi'(t) = 0$, $\lim_{t\to+\infty} t\varphi'(t) = 0$, that is $\varphi'(t)$ is monotonically decreasing to 0 at least as fast as $1/t$;
(3a) $\varphi(0) = 0$, $\lim_{t\to+\infty} \varphi(t) = B > 0$
(3b) $\varphi(0) = 0$, $\lim_{t\to+\infty} \varphi(t) = B \geqslant 0$
(4) $\lim_{t\to-\infty} \varphi(t) = -\infty$
(5) $\lim_{t\to+\infty} \varphi''(t) = 0$

Observation 1. Since $\varphi(0) = 0$ and for property (1) $\varphi(t)$ is an increasing function, we have $\varphi(t) > 0$ when $t > 0$ and $\varphi(t) < 0$ when $t < 0$.

In the sequel we introduce two classes of filled functions. In the first class we use functions $\varphi(t)$ satisfying propeties (1), (2), (3a) and some examples are:

- $\varphi(t) = \arctan(t)$
- $\varphi(t) = t/(1+t)$

The second class requires functions $\varphi(t)$ satisfying properties (1), (2), (3b), (4), (5) and some examples are:

- $\varphi(t) = \min\{t, 0\}^3$
- $\varphi(t) = 1 - e^{-t}$

The functions $\eta(t)$ and $\varphi(t)$ play two different important roles in defining filled functions:

- by setting $t = \tau[f(x) - f(x_1^*) + \rho]$ in $\varphi(t)$, we obtain a term that, for sufficiently large values of τ, can filter stationary points $\hat{x}$ such that $f(\hat{x}) < f(x_1^*)$; in fact in the region $\{x \in R^n: f(x) \geqslant f(x_1^*)\}$ the behaviour of $\varphi(\tau[f(x) - f(x_1^*) + \rho])$ becomes more and more flat for increasing values of τ; while in the region $\{x \in R^n: f(x) < f(x_1^*)\}$ we have that $\varphi(\tau[f(x) - f(x_1^*) + \rho]) < 0$ and it has a 'sufficient slop';
- by setting $t = \frac{1}{2}\|x - x_0\|^2$ in $\eta(t)$, we obtain a term $\eta(\frac{1}{2}\|x - x_0\|^2)$ that is able to enforce the global convexity of filled functions and to guarantee a 'sufficient slop' in the region where $\varphi(\tau[f(x) - f(x_1^*) + \rho])$ is almost flat.

2.1. A CONTINUOUSLY DIFFERENTIABLE GLOBALLY CONVEXIZED FILLED FUNCTION

Given a $x_0 \in R^n$, a local minimum x_1^* of problem (1) and two parameters, $\tau \geqslant 1$ and $\rho > 0$, we introduce the following new class of continuously differentiable filled functions:

$$W(x, \tau, \rho) = \eta\left(\frac{1}{2}\,\|x - x_0\|^2\right)\varphi(\tau[f(x) - f(x_1^*) + \rho]) .$$

The gradient of $W(x, \tau, \rho)$ is:

$$\nabla W(x, \tau, \rho) = (x - x_0)\eta'\left(\frac{1}{2}\,\|x - x_0\|^2\right)\varphi(\tau[f(x) - f(x_1^*) + \rho]) + \tau\eta\left(\frac{1}{2}\,\|x - x_0\|^2\right)\nabla f(x)\varphi'(\tau[f(x) - f(x_1^*) + \rho]) . \quad (2)$$

For any $\tilde{x} \in R^n$, we denote the level set of $W(x, \tau, \rho)$ by

$$\mathcal{L}_{\mathcal{W}}(\tilde{x}, \tau, \rho) = \{x \in R^n : W(x, \tau, \rho) \leqslant W(\tilde{x}, \tau, \rho)\} .$$

In the sequel of this subsection we suppose that the following conditions hold:

ASSUMPTION 2.1.
(i) $\eta(t)$ satisfies properties (1) and (2);
(ii) $\varphi(t)$ satisfies properties (1), (2) and (3a);
(iii) ρ satisfies:

$$0 < \rho < f(x_1^*) - f(x^*) \quad (3)$$

where x^* is a global minimizer of $f(x)$.

First we state that $W(\tilde{x}, \tau, \rho)$ is globally convex and this implies that the level set $\mathcal{L}_{\mathcal{W}}(\tilde{x}, \tau, \rho)$ is compact. Moreover we prove that the sets $\mathcal{L}_{\mathcal{W}}(\tilde{x}, \tau, \rho)$, for all $\tau \geqslant 1$ and for all $\rho > 0$ are contained in a compact set Δ.

THEOREM 1. *For every τ, ρ the function $W(x, \tau, \rho)$ is globally convex and hence the level set $\mathcal{L}_{\mathcal{W}}(\tilde{x}, \tau, \rho)$ is compact.*

Furthermore there exists a compact set Δ such that

$$\mathcal{L}_{\mathcal{W}}(\tilde{x}, \tau, \rho) \subseteq \Delta \quad (4)$$

for all $\tau \geqslant 1$ and $\rho > 0$.

Proof. First we consider the global convexity of the function. Since $f(x) \to \infty$ when $\|x\| \to \infty$, then even $f(x) - f(x_1^*) + \rho \to +\infty$ when $\|x\| \to \infty$; it follows from the properties of $\varphi(t)$ that $\lim_{\|x\| \to +\infty} \varphi(\tau[f(x) - f(x_1^*) + \rho]) = B > 0$. Moreover we have from the properties of $\eta(t)$:

$$\eta(t) = \int_0^t \eta'(s)\,\mathrm{d}s \geqslant \int_0^t \tilde{a}\,\mathrm{d}s = \tilde{a}t \to +\infty \quad \text{when} \quad t \to +\infty .$$

Thus

$$\lim_{\|x\| \to +\infty} W(x, \tau, \rho) = +\infty$$

This proves the global convexity of $W(x, \tau, \rho)$.

Now we turn to prove the second part of the thesis. Properties (1) and (3) of the function φ imply that for all x and for all $\tau \geqslant 1$ we have:

$$\varphi(\tau[f(x) - f(x_1^*) + \rho]) \leqslant B \tag{5}$$

$$\varphi(\tau[f(x) - f(x_1^*) + \rho]) \geqslant \varphi([f(x) - f(x_1^*)]) \tag{6}$$

and then

$$W(x, \tau, \rho) \leqslant \eta\left(\frac{1}{2}\|x - x_0\|^2\right)B \tag{7}$$

$$W(x, \tau, \rho) \geqslant \eta\left(\frac{1}{2}\|x - x_0\|^2\right)\varphi([f(x) - f(x_1^*)]) \tag{8}$$

Now, by using (7) and (8), we have:

$$\mathscr{L}_{\mathscr{W}}(\tilde{x}, \tau, \rho) \subseteq \left\{x \in R^n : \eta\left(\frac{1}{2}\|x - x_0\|^2\right)\varphi(f(x) - f(x_1^*)) \leqslant \eta\left(\frac{1}{2}\|\tilde{x} - x_0\|^2\right)B\right\} \tag{9}$$

which completes the proof by setting

$$\Delta = \left\{x \in R^n : \eta\left(\frac{1}{2}\|x - x_0\|^2\right)\varphi(f(x) - f(x_1^*)) \leqslant \eta\left(\frac{1}{2}\|\tilde{x} - x_0\|^2\right)B\right\}. \qquad \square$$

Now we study the nature of the point x_0.

THEOREM 2. *If* $f(x_0) \geqslant f(x_1^*)$, *then the prefixed point* x_0 *is an isolated local minimizer of* $W(x, \tau, \rho)$.

Proof. The continuity of $f(x)$ on Δ, implies the existence of a neighbourhood of x_0, called $N(x_0)$, such that $f(x) \geqslant f(x_1^*)$ holds for $x \in N(x_0)$. Therefore we have in this neighbourhood

$$W(x_0, \tau, \rho) = 0 < \eta\left(\frac{1}{2}\|x - x_0\|^2\right)\varphi(\tau[f(x) - f(x_1^*) + \rho]) = W(x, \tau, \rho)$$

This shows that x_0 is an isolated local minimizer of $W(x, \tau, \rho)$, which is what we wanted to prove. $\square$

The next result characterizes the stationary points $\hat{x}$ of $W(x, \tau, \rho)$. First we split the level set $\mathscr{L}_{\mathscr{W}}(\tilde{x}, \tau, \rho)$ in two subsets:

$$S_1 = \{x \in R^n : f(x) \geqslant f(x_1^*), x \in \mathscr{L}_{\mathscr{W}}(\tilde{x}, \tau, \rho)\} \tag{10}$$

$$S_2 = \{x \in R^n : f(x) < f(x_1^*), x \in \mathscr{L}_{\mathscr{W}}(\tilde{x}, \tau, \rho)\} \tag{11}$$

THEOREM 3. *(a) There exists a* $\bar{\tau} \geq 1$ *such that for all* $\tau \geq \bar{\tau}$ *we have that the function* $W(x, \tau, \rho)$ *has no stationary points in the region* S_1 *except the prefixed point* x_0.

(b) If x_1^* *is not a global minimum of f(x) and* ρ *satisfies (3) then the function* $W(x, \tau, \rho)$ *has a minimizer in the region* S_2.

Proof. (a) Recalling the expression (2) of the gradient of $W(x, \tau, \rho)$ we have that a stationary point $\hat{x}$ of $W(x, \tau, \rho)$ must satisfy

$$\begin{aligned} &\|\hat{x} - x_0\|\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\varphi(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \\ &\quad = \tau\eta\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\|\nabla f(\hat{x})\|\varphi'(\tau[f(\hat{x}) - f(x_1^*) + \rho]) . \end{aligned} \tag{12}$$

Suppose $\hat{x} \in S_1$. This implies that $f(\hat{x}) \geq f(x_1^*)$.

Since x_0 is an isolated local mimimizer of $W(x, \tau, \rho)$, it is possible to construct a neighbourhood $B(x_0, \epsilon_0)$ with no stationary point of $W(x, \tau, \rho)$, so we have $\|\hat{x} - x_0\| > \epsilon_0$. Moreover the properties of $\varphi(t)$ and $\eta(t)$ imply that:

$$\epsilon_0 \tilde{a} \varphi(\tau\rho) \leq \|\hat{x} - x_0\|\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\varphi(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \tag{13}$$

and

$$\tau\eta\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\|\nabla f(\hat{x})\|\varphi'(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \leq \tau\eta\left(\frac{1}{2}D'^2\right)L\varphi'(\tau\rho) \tag{14}$$

where $D' = \max_{x \in \Delta}\|x - x_0\|$ and $L = \max_{x \in \Delta}\|\nabla f(x)\|$.

Thus if (12) holds and $x \in S_1$ we should have:

$$\epsilon_0 \tilde{a} \varphi(\tau\rho) \leq \tau\eta\left(\frac{1}{2}D'^2\right)L\varphi'(\tau\rho) . \tag{15}$$

Now, recalling again properties (2) and (3) of $\varphi(t)$, we have

$$\lim_{\tau \to +\infty} \epsilon_0 \tilde{a} \varphi(\tau\rho) = \epsilon_0 \tilde{a} B$$

$$\lim_{\tau \to +\infty} \tau\eta\left(\frac{1}{2}D'^2\right)L\varphi'(\tau\rho) = \lim_{\tau \to +\infty} \frac{\tau\rho\eta\left(\frac{1}{2}D'^2\right)L\varphi'(\tau\rho)}{\rho} = 0$$

This means we can always find a value $\bar{\tau}$ such that for all $\tau \geq \bar{\tau}$ we have:

$$\epsilon_0 \tilde{a} \varphi(\tau\rho) > \epsilon_0 \tilde{a} \frac{B}{2}$$

$$\tau\eta\left(\frac{1}{2}D'^2\right)L\varphi'(\tau\rho) < \epsilon_0 \tilde{a} \frac{B}{2}$$

and these relations contradict (15). This shows that every stationary point different from x_0 cannot belong to S_1.

(b) Since $W(x, \tau, \rho)$ is continuous function in the compact level set $\mathscr{L}_{\mathscr{W}}(\tilde{x}, \tau, \rho)$, it has a global minimum $\bar{x}$. Let x^* be a global minimum of $f(x)$. By using (3), we have

$f(x^*) < f(x_1^*) - \rho$. Moreover Observation 1 implies $\varphi(\tau[f(x^*) - f(x_1^*) + \rho]) < 0$. From the definition of $W(x, \tau, \rho)$ we have that $W(x^*, \tau, \rho) < 0$, and hence

$$W(\bar{x}, \tau, \rho) \leq W(x^*, \tau, \rho) < 0 .$$

This implies that $\varphi(\tau[f(\bar{x}) - f(x_1^*) + \rho]) < 0$ which implies in turn that $f(\bar{x}) < f(x_1^*)$, that is $\bar{x} \in S_2$. □

Theorems 1 and 3 show that for suitable values of τ an unconstrained minimization of the function $W(x, \tau, \rho)$ is able to locate either a point $\hat{x}$ such that $f(\hat{x}) < f(x_1^*)$ or the prefixed point x_0. This feature can be useful in the context of global optimization methods to escape from the local minima. However, an efficient use of the filled function $W(x, \tau, \rho)$ (or the use of filled functions proposed in Ge and Qin (1990)) requires to deal with the following two difficulties:

- a good choice of the parameter τ;
- the presence of the local minimum point x_0.

In fact too small values of the parameter τ could not satisfy the requirement $\tau \geq \bar{\tau}$ (see Theorem 3), while too large values of the parameter τ could make the filled function difficult to numerically minimize. Therefore updating procedures should be used to identify suitable values for the parameter τ.

As regard the presence of the local minimum x_0, it is desirable to avoid the local minimizations of $W(x, \tau, \rho)$ be entrapped by this point. This could be done by using ‘proper’ procedures for changing the starting point of the local minimization or for changing the point x_0.

The definition of these techniques for $W(x, \tau, \rho)$ could not be easy, in fact it requires to be able to foresee the combined effects of simultaneously changing τ and x_0. For this reason in the next subsection we introduce a new class of filled functions with an additive structure, which keeps apart the two terms, $\eta(\frac{1}{2}\|x - x_0\|^2)$ and $\varphi(\tau[f(x) - f(x_1^*) + \rho])$, and hence keeps apart the effects of changing τ and x_0.

2.2. A NEW FILLED FUNCTION WITH ADDITIVE STRUCTURE

We introduce the following class of filled functions:

$$V(x, \tau, \rho) = \eta\left(\frac{1}{2}\|x - x_0\|^2\right) + \varphi(\tau[f(x) - f(x_1^*) + \rho])$$

with gradient given by:

$$\nabla V(x, \tau, \rho) = (x - x_0)\eta'\left(\frac{1}{2}\|x - x_0\|^2\right) + \tau\nabla f(x)\varphi'(\tau[f(x) - f(x_1^*) + \rho]) \qquad (16)$$

For every $\tilde{x} \in R^n$ we denote the level set of $V(x, \tau, \rho)$ by

$$\mathscr{L}_V(\tilde{x}, \tau, \rho) = \{x \in R^n : V(x, \tau, \rho) \leq V(\tilde{x}, \tau, \rho)\} .$$

To state the properties of $V(x, \tau, \rho)$ we require that the following assumptions hold.

ASSUMPTIONS 2.2
(i) $\eta(t)$ satisfies properties (1) and (2);
(ii) $\varphi(t)$ satisfies properties (1), (2), (3b), (4) and (5);
(iii) ρ satisfies:

$$0 < \rho < f(x_1^*) - f(x^*) \tag{17}$$

where x^* is a global minimizer of $f(x)$.

As for $W(x, \tau, \rho)$, we can prove that $V(x, \tau, \rho)$ is globally convex and that the sets $\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho)$, for all $\tau > 0$ and for all $\rho > 0$, are contained in a compact set Δ.

THEOREM 4. *For every $\tau, \rho > 0$ the function $V(x, \tau, \rho)$ is globally convex and hence the level set $\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho)$ is compact*

Furthermore there exists a compact set Δ such that

$$\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho) \subseteq \Delta \tag{18}$$

for all $\tau > 0$ and $\rho > 0$.

Proof. First we consider the global convexity of the function. When $\|x\| \to +\infty$ we have $f(x) \to +\infty$, thus $f(x) - f(x_1^*) + \rho \to +\infty$. It follows from property (3) of $\varphi(t)$ that

$$\lim_{\|x\| \to +\infty} \varphi(\tau[f(x) - f(x_1^*) + \rho]) = B .$$

Moreover we have from the properties of $\eta(t)$

$$\eta(t) = \int_0^t \eta'(s)\, ds \geqslant \int_0^t \tilde{a}\, ds = \tilde{a}t \to +\infty \quad \text{when} \quad t \to +\infty$$

so

$$\lim_{\|x\| \to +\infty} \eta\left(\frac{1}{2}\|x - x_0\|^2\right) = +\infty$$

This proves the global convexity of $V(x, \tau, \rho)$.

Now we turn to prove the second part of the thesis. Properties (1) and (3) of the function φ imply that $\varphi(t) \leqslant B$ for all t and, hence, we have:

$$\begin{aligned} V(x, \tau, \rho) &= \eta\left(\frac{1}{2}\|x - x_0\|^2\right) + \varphi(\tau[f(x) - f(x_1^*) + \rho]) \\ &\leqslant \eta\left(\frac{1}{2}\|x - x_0\|^2\right) + B . \end{aligned} \tag{19}$$

Now, by using (19), we have:

$$\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho) \subseteq \left\{x \in R^n : \eta\left(\frac{1}{2}\|x - x_0\|^2\right) + \varphi(\tau[f(x) - f(x_1^*) + \rho]) \leqslant \delta\right\} \tag{20}$$

where $\delta = \eta(\frac{1}{2}\|\tilde{x} - x_0\|^2) + B$.

By the definition of $\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho)$ we have:

$$\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho) \subseteq \Delta_1 \cup \Delta_2 \tag{21}$$

where

$$\Delta_1 = \left\{ x \in R^n \colon \eta\left(\frac{1}{2}\|x - x_0\|^2\right) + \varphi(\tau[f(x) - f(x_1^*) + \rho]) \leqslant \delta \right., \left. \varphi(\tau[f(x) - f(x_1^*) + \rho]) < 0 \right\}$$

and

$$\Delta_2 = \left\{ x \in R^n \colon \eta\left(\frac{1}{2}\|x - x_0\|^2\right) + \varphi(\tau[f(x) - f(x_1^*) + \rho]) \leqslant \delta \right., \left. \varphi(\tau[f(x) - f(x_1^*) + \rho]) \geqslant 0 \right\}.$$

Recalling observation 1 we note that $\varphi(t) < 0$ implies $t < 0$ and then

$$\Delta_1 \subseteq \{x \in R^n \colon f(x) \leqslant f(x_1^*)\} \tag{22}$$

where $\{x \in R^n \colon f(x) \leqslant f(x_1^*) - \rho\}$ is compact for the assumption of global convexity of $f(x)$.

As regards Δ_2 we note

$$\Delta_2 \subseteq \left\{ x \in R^n \colon \eta\left(\frac{1}{2}\|x - x_0\|^2\right) \leqslant \delta \right\} \tag{23}$$

where the set on the right side of the inclusion is clearly compact.

Finally the (18) follows from (20), (21), (22), (23) by setting

$$\Delta = \{x \in R^n \colon f(x) \leqslant f(x_1^*)\} \cup \left\{ x \in R^n \colon \eta\left(\frac{1}{2}\|x - x_0\|^2\right) \leqslant \delta \right\}. \qquad \square$$

Also for this class of filled functions we can characterize the stationary points $\hat{x}$ of $V(x, \tau, \rho)$. To do this we split the level set $\mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho)$ in two subsets:

$$S_1 = \{x \in R^n \colon f(x) \geqslant f(x_1^*), x \in \mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho)\} \tag{24}$$

$$S_2 = \{x \in R^n \colon f(x) < f(x_1^*), x \in \mathscr{L}_{\mathscr{V}}(\tilde{x}, \tau, \rho)\}. \tag{25}$$

THEOREM 5. *There exist values* $\bar{\tau} > 0$ *and* $\bar{\epsilon} > 0$ *such that for all* $\tau \geqslant \bar{\tau}$ *and for all* $\epsilon \in (0, \bar{\epsilon}]$ *we have*:

(a) the function $V(x, \tau, \rho)$ *has no stationary points in the region* S_1, *except in the neighbourhood* $B(x_0, \epsilon)$ *of* x_0, *where an isolated local minimum point can exist*;

(b) if x_1^* *is not a global minimum of* $f(x)$ *and* ρ *satisfies* (17), *then all the global minimum points of the function* $V(x, \tau, \rho)$ *are in the region* S_2.

Proof. (a) Recalling the expression (16) of the gradient of $V(x, \tau, \rho)$ we note that a stationary point $x_\tau \in S_1$ of $V(x, \tau, \rho)$ must satisfy

$$\|x_\tau - x_0\|\eta'\left(\frac{1}{2}\|x_\tau - x_0\|^2\right) = \tau\|\nabla f(x_\tau)\|\varphi'(\tau[f(x_\tau) - f(x_1^*) + \rho]) . \tag{26}$$

Since $x_\tau \in S_1$, property (2) of $\varphi(t)$ yields:

$$\tau\|\nabla f(x_\tau)\|\varphi'(\tau[f(x_\tau) - f(x_1^*) + \rho]) \leqslant \tau L\varphi'(\tau\rho) \tag{27}$$

where $L = \max_{x\in\Delta}\|\nabla f(x)\|$.

Property (2) of $\eta(t)$ gives:

$$\|x_\tau - x_0\|\eta'\left(\frac{1}{2}\|x_\tau - x_0\|^2\right) \geqslant \|x_\tau - x_0\|\tilde{a} . \tag{28}$$

By (26), (27) and (28) we obtain

$$\|x_\tau - x_0\| \leqslant \frac{\tau L\varphi'(\tau\rho)}{\tilde{a}} . \tag{29}$$

Property (2) of $\varphi(t)$ implies that we can find a $\hat{\tau}$ such that, for all $\tau \geqslant \hat{\tau}$, $x_\tau \in B(x_0, \epsilon)$.

We recall the expression of $\nabla^2 V(x, \tau, \rho)$:

$$\begin{aligned}\nabla^2 V(x, \tau, \rho) = \eta'\left(\frac{1}{2}\|x - x_0\|^2\right)I + \eta''\left(\frac{1}{2}\|x - x_0\|^2\right)(x - x_0)(x - x_0)^T \\ + \tau\varphi'(\tau[f(x) - f(x_1^*) + \rho])\nabla^2 f(x) \\ + \tau^2\varphi''(\tau[f(x) - f(x_1^*) + \rho])\nabla f(x)f(x)^T .\end{aligned} \tag{30}$$

By using the structure of $\nabla^2 V(x, \tau, \rho)$ and recalling property (2) of $\eta(t)$ and properties (2) and (5) of $\varphi(t)$, we can observe that there exist values $\tilde{\tau} \geqslant \hat{\tau}$ and $\bar{\epsilon} > 0$ such that for all $\tau \geqslant \tilde{\tau}$, for all $\epsilon \in (0, \bar{\epsilon}]$ and for all $x \in B(x_0, \epsilon)$ the hessian matrix $\nabla^2 V(x, \tau, \rho)$ is positive definite. Therefore we have that x_τ is an isolated local minimum point and that there are no other stationary points in $B(x_0, \epsilon)$.

(b) Let x^* be a global minimum of $f(x)$. By using (17), we have $f(x^*) < f(x_1^*) - \rho$. Moreover Observation 1 implies $\varphi(\tau[f(x^*) - f(x_1^*) + \rho]) < 0$. It follows from property (4) of $\varphi(t)(\varphi(t)) \to -\infty$ when $t \to -\infty$) that there is a value $\bar{\tau} \geqslant \hat{\tau}$ such that, for all $\tau \geqslant \bar{\tau}$, $V(x^*, \tau, \rho) < 0$. Since $V(x, \tau, \rho)$ is a continuous function in the compact level set $\mathscr{L}_\gamma(\tilde{x}, \tau, \rho)$, it has a global minimum $\bar{x}$, which clearly satisfies

$$V(\bar{x}, \tau, \rho) \leqslant V(x^*, \tau, \rho) < 0 .$$

for all $\tau \geqslant \bar{\tau}$.

This implies that $\varphi(\tau[f(\bar{x}) - f(x_1^*) + \rho]) < 0$, which implies in turn that $f(\bar{x}) < f(x_1^*)$, that is $\bar{x} \in S_2$. □

We note that the function $V(x, \tau, \rho)$ has theoretical properties similar to the ones of $W(x, \tau, \rho)$. In fact, for suitable values of τ, an unconstrained mimimization of the function $V(x, \tau, \rho)$ is able to locate either a point $\hat{x}$ such that $f(\hat{x}) < f(x_1^*)$ or a point x_τ sufficiently close to x_0. Thus the only difference is that we do not exactly know where x_τ is located. Anyway, by the proof of the previous theorem, we can note that, for increasing values of τ, x_τ tends to x_0.

Finally we can also observe that Theorem 4 and Theorem 5 still hold by setting $\rho = 0$, if the function $\varphi(t)$ satisfies the additional assumption $\varphi'(0) = 0$.

3. A class of filled functions for box constrained optimization problems

In this section we consider the following optimization problem:

$$\min_{x \in \mathcal{F}} f(x) \tag{31}$$

where $\mathcal{F} = \{x | l_i \leqslant x_i \leqslant u_i, i = 1, \ldots, n\}$ with $l_i, u_i \in \Re$.

A stationary point for Problem (31) is a point $x^* \in \mathcal{F}$ which satisfies the following necessary conditions:

$$\frac{\partial f}{\partial x_i} \geqslant 0 \quad \text{if } x_i^* = l_i$$

$$\frac{\partial f}{\partial x_i} \leqslant 0 \quad \text{if } x_i^* = u_i$$

$$\frac{\partial f}{\partial x_i} = 0 \quad \text{if } l_i < x_i^* < u_i \,.$$

The box constrained minimization problems are more representative than unconstrained ones because they can represent a wider class of real applications. For this reason the aim of the next two subsections is to extend our approach to this class of problems.

The presence of the box constraints makes harder the local minimization process, but, roughly speaking, is useful to define the properties of the filled functions. In fact, since the feasible set $\mathcal{F}$ is compact, we do not need to ensure any compactness property of the level sets of the filled functions.

3.1. A FIRST FILLED FUNCTION FOR BOX CONSTRAINED OPTIMIZATION PROBLEMS

We want to show that the function $W(x, \tau, \rho)$ preserves its properties in the case of problems with box constraints. We suppose the point x_0 strictly interior to $\mathcal{F}$, that is $l_i < x_{0i} < u_i$, $i = 1, \ldots, n$.

We suppose that Assumptions 2.1 hold.

The next theorem extends the analysis of Theorem 3 to the constrained stationary points of Problem (31). For this aim we split the feasible set $\mathcal{F}$ in two subsets:

$$S_1 = \{x \in R^n \colon f(x) \geqslant f(x_1^*), x \in \mathcal{F}\} \tag{33}$$

$$S_2 = \{x \in R^n \colon f(x) < f(x_1^*), x \in \mathcal{F}\} \tag{34}$$

THEOREM 6. *(a) There exists a* $\bar{\tau} \geqslant 1$ *such that for all* $\tau \geqslant \bar{\tau}$ *we have that the function* $W(x, \tau, \rho)$ *has no stationary points in the region* S_1 *except the prefixed point* x_0.
(b) If x_1^* *is not a global minimum of* $f(x)$ *and* ρ *satisfies (3), then the function* $W(x, \tau, \rho)$ *has a minimizer in the region* S_2.

Proof. (a) Let $\hat{x}$ be a stationary point of the function $W(x, \tau, \rho)$. It must satisfy conditions (32). We have two cases:

(1) The point $\hat{x}$ is strictly interior to the feasible set, namely $l_i < \hat{x}_i < u_i$ for all $i = 1, \dots, n$. Therefore $\hat{x}$ is an unconstrained stationary point and then Theorem 3 ensures that exists a value $\hat{\tau} \geqslant 1$ such that for all $\tau \geqslant \hat{\tau}$ the function $W(x, \tau, \rho)$ has no unconstrained stationary points in the region S_1 except the prefixed point x_0.

(2) There exists at least an index j such that either $\hat{x}_j = l_j$ or $\hat{x}_j = u_j$. If $\hat{x}_j = l_j$ condition (32) implies

$$\frac{\partial W(\hat{x}, \tau, \rho)}{\partial x_j} \geqslant 0$$

that is

$$(l_j - x_{0j})\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\varphi(\tau[f(\hat{x}) - f(x_1^*) + \rho]) + \tau\eta\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\frac{\partial f(\hat{x})}{\partial x_j}\varphi'(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \geqslant 0 . \tag{35}$$

We suppose $\hat{x} \in S_1$, then $\tau[f(\hat{x}) - f(x_1^*) + \rho] > 0$. This implies $\varphi(\tau[f(\hat{x}) - f(x_1^*) + \rho]) > 0$. Furthermore recalling that $l_j - x_{0j} < 0$ and that $\eta'(\frac{1}{2}\|\hat{x} - x_0\|^2) > 0$, we have

$$0 \leqslant (x_{0j} - l_j)\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\varphi(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \leqslant \tau\eta\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\frac{\partial f(\hat{x})}{\partial x_j}\varphi'(\tau[f(\hat{x}) - f(x_1^*) + \rho]) . \tag{36}$$

By using property (2) of $\eta(t)$ and property (1) $\varphi(t)$, since $\tau \geqslant 1$ we have:

$$\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\varphi(\tau[f(\hat{x}) - f(x_1^*) + \tau\rho]) \geqslant \tilde{a}\varphi(\rho) > 0 . \tag{37}$$

Moreover property (2) of $\eta(t)$, property (2) of $\varphi(t)$ and $\tau \geqslant 1$ imply

$$\tau\eta\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right)\frac{\partial f(\hat{x})}{\partial x_j}\varphi'(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \leqslant \tau\eta\left(\frac{1}{2}\|l - u\|^2\right)L\varphi'(\tau\rho) \tag{38}$$

where $L = \max_{x \in \mathscr{F}}\|\nabla f(x)\|$.

Thus if (36) holds, we should have:

$$(x_{0j} - l_j)\tilde{a}\varphi(\rho) \leqslant \tau\eta\left(\frac{1}{2}\|l - u\|^2\right)L\varphi'(\tau\rho) . \tag{39}$$

Now, recalling again property (2) of $\varphi(t)$ we have

$$\lim_{\tau\to+\infty} \tau\varphi'(\tau\rho) = \lim_{\tau\to+\infty} \frac{\tau\rho\varphi'(\tau\rho)}{\rho} = 0\ . \tag{40}$$

This means we can always find a value $\tilde{\tau} \geqslant \hat{\tau}$ such that for all $\tau \geqslant \tilde{\tau}$ we have:

$$\tau\eta\left(\frac{1}{2}\|l-u\|^2\right)L\varphi'(\tau\rho) < (x_0 - l_j)\tilde{a}\varphi(\rho)$$

and this relation contradicts (39) and hence (35).

Instead if $\hat{x}_j = u_j$ condition (32) implies

$$\frac{\partial W(\hat{x}, \tau, \rho)}{\partial x_j} \leqslant 0$$

that is

$$(u_j - x_{0j})\eta'\left(\frac{1}{2}\|\hat{x}-x_0\|^2\right)\varphi(\tau[f(\hat{x}) - f(x_1^*) + \rho])$$
$$+\ \tau\eta\left(\frac{1}{2}\|\hat{x}-x_0\|^2\right)\frac{\partial f(\hat{x})}{\partial x_j}\varphi'(\tau[f(\hat{x}) - f(x_1^*) + \rho]) \leqslant 0\ . \tag{41}$$

By using again (37) and (38) we can write

$$(u_j - x_{0j})\tilde{a}\varphi(\rho) \leqslant \tau\eta\left(\frac{1}{2}\ \|l-u\|^2\right)L\varphi'(\tau\rho)\ . \tag{42}$$

Recalling (40) we have that we can always find a value $\bar{\tau} \geqslant \tilde{\tau}$ such that for all $\tau \geqslant \bar{\tau}$ we have:

$$\tau\eta\left(\frac{1}{2}\|l-u\|^2\right)L\varphi'(\tau\rho) < (u_j - x_{0j})\tilde{a}\varphi(\rho)$$

and this relation contradicts (42) and hence (41).

Finally we can conclude that there exists a value $\bar{\tau} \geqslant 1$ such that for all $\tau \geqslant \bar{\tau}$ the function $W(x, \tau, \rho)$ has no stationary points in the region S_1 except the prefixed point x_0.

(*b*) The proof of this point is the same of point (b) of Theorem 3. □

3.2. A FILLED FUNCTION WITH ADDITIVE STRUCTURE FOR BOX CONSTRAINED OPTIMIZATION PROBLEMS

In this subsection we study the properties of the function $V(x, \tau, \rho)$ in the case of problems with box constraints. We suppose that the point x_0 is inner to $\mathscr{F}$, that is $l_i < x_{0i} < u_i,\ i = 1, \ldots, n$.

ASSUMPTION 3.1. (i) $\eta(t)$ satisfies properties (1) and (2);
(ii) $\varphi(t)$ satisfies properties (1), (2), (3b), (5) and
(4b) $|\varphi(t)| \geqslant \theta\eta(\frac{1}{2}\|u-l\|^2)$, with $\theta > 1$ when $t \to -\infty$;

(iii) ρ satisfies:

$$0 < \rho < f(x_1^*) - f(x^*) \tag{43}$$

where x^* is a global minimizer of $f(x)$.

We note that by scaling properly the function $\eta(t)$, all the samples of function $\varphi(t)$ reported in Section 2 satisfy properties (1), (2), (3b), (4b).

We use again the sets S_1 and S_2 defined by (33) and (34).

THEOREM 7. *There exist values $\bar{\tau} > 0$ and $\bar{\epsilon} > 0$ such that for all $\tau \geqslant \bar{\tau}$ and for all $\epsilon \in (0, \bar{\epsilon}]$ we have*:

(a) the function $V(x, \tau, \rho)$ has no stationary points in the region S_1, except in the neighbourhood $B(x_0, \epsilon)$ of x_0, where an isolated local minimum point can exist;

(b) if x_1^* is not a global minimun of $f(x)$ and ρ satisfies (43), *then all the global minimum points of the function $V(x, \tau, \rho)$ are in the region S_2.*

Proof. (a) Let $\hat{x}$ be a stationary point of the function $V(x, \tau, \rho)$. It must satisfy conditons (32). We have two cases:

(1) the point $\hat{x}$ is an unconstrained stationary point ($l_i < \hat{x}_i < u_i$ for all $i = 1, \ldots, n$), then Theorem 5 implies that there exist values $\hat{\tau} > 0$ and $\bar{\epsilon} > 0$ such that for all $\tau \geqslant \hat{\tau}$ and for all $\epsilon \in (0, \bar{\epsilon}]$ the function $V(x, \tau, \rho)$ has no unconstrained stationary points in the region S_1, except in the neighbourhood $B(x_0, \epsilon)$ of x_0.

(2) The point $\hat{x}$ is a constrained stationary point, namely there exists at least an index j such that either $\hat{x}_j = l_j$ or $\hat{x}_j = u_j$.

If $\hat{x}_j = l_j$ condition (32) implies

$$\frac{\partial V(\hat{x}, \tau, \rho)}{\partial x_j} \geqslant 0$$

that is

$$(l_j - x_{0j})\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right) + \tau\varphi'(\tau[f(\hat{x}) - f_1^*) + \rho])\frac{\partial f(\hat{x})}{\partial x_j} \geqslant 0\,. \tag{44}$$

Recalling that $l_j - x_{0j} < 0$ and that $\eta'(\frac{1}{2}\|\hat{x} - x_0\|^2) > 0$ we have

$$(l_j - x_{0j})\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right) < 0\,. \tag{45}$$

We suppose $\hat{x} \in S_1$, then $\tau[f(\hat{x}) - f(x_1^*) + \rho] > 0$. Inequality (44) can be rewritten as:

$$(x_{0j} - l_j)\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right) \leqslant \tau\varphi'(\tau[f(x_\tau) - f(x_1^*) + \rho])\frac{\partial f(\hat{x})}{\partial x_j}\,. \tag{46}$$

By using property (2) of $\eta(t)$ and properties of $\varphi(t)$, we have:

$$(x_{0j} - l_j)\tilde{a} \leqslant \tau\varphi'(\tau\rho)L \tag{47}$$

where $L = \max_{x\in\mathscr{F}} \|\nabla f(x)\|$. Recalling property (2) of $\varphi(t)$ there exists a value $\tilde{\tau} \geqslant \hat{\tau}$ such that for all $\tau \geqslant \tilde{\tau}$ we have

$$(x_{0j} - l_j)\tilde{a} > \tau\varphi'(\tau\rho)L \tag{48}$$

which contradicts (47) and then (46).

Instead if $\hat{x}_j = u_j$ condition (32) implies

$$\frac{\partial V(\hat{x}, \tau, \rho)}{\partial x_j} \leqslant 0$$

that is

$$(u_j - x_{0j})\eta'\left(\frac{1}{2}\|\hat{x} - x_0\|^2\right) + \tau\varphi'(\tau[f(x_\tau) - f_1^*) + \rho])\frac{\partial f(\hat{x})}{\partial x_j} \leqslant 0\,. \tag{49}$$

By using property (2) of $\eta(t)$ and properties of $\varphi(t)$, we have:

$$(u_j - x_{0j})\tilde{a} \leqslant \tau\varphi'(\tau\rho)L \tag{50}$$

where $L = \max_{x\in\mathscr{F}} \|\nabla f(x)\|$. Recalling property (2) of $\varphi(t)$ there exists a value $\tau' \geqslant \tilde{\tau}$ such that for all $\tau \geqslant \tau'$ we have

$$(u_j - x_{0j})\tilde{a} > \tau\varphi'(\tau\rho)L \tag{51}$$

which contradicts (50) and then (49).

Finally we can conclude that there exists a value $\tau' \geqslant 1$ such that for all $\tau \geqslant \tau'$ the function $V(x, \tau, \rho)$ has no stationary points in the region S_1, except in the neighbourhood $B(x_0, \epsilon)$ of x_0, where an isolated local minimum point can exist.

(b) Let x^* be a global minimum of $f(x)$. By using (3), we have $f(x^*) < f(x_1^*) - \rho$. Moreover Observation 1 implies $\varphi(\tau[f(x^*) - f(x_1^*) + \rho]) < 0$. We consider the expression of $V(x^*, \tau, \rho)$:

$$V(x^*, \tau, \rho) = \eta\left(\frac{1}{2}\|x^* - x_0\|^2\right) + \varphi(\tau[f(x^*) - f(x_1^*) + \rho])\,.$$

For the properties of $\eta(t)$ we have

$$\eta\left(\frac{1}{2}\|x^* - x_0\|^2\right) \leqslant \eta\left(\frac{1}{2}\|u - l\|^2\right).$$

It follows from property (4b) of $\varphi(t)(|\varphi(t)| \geqslant \theta\eta(\frac{1}{2}\|u - l\|^2)$, with $\theta > 1$ when $t \to -\infty$) that exists a value $\tau^* > 0$ such that for all $\tau \geqslant \tau^*$ we have $V(x^*, \tau, \rho) < 0$. Since $V(x, \tau, \rho)$ is a continuous function in the compact set $\mathscr{F}$, it has a global minimum $\bar{x}$, which clearly satisfies

$$V(\bar{x}, \tau, \rho) \leqslant V(x^*, \tau, \rho) < 0 .$$

for all $\tau \geqslant \tau^*$.

This implies that $\varphi(\tau[f(\bar{x}) - f(x_1^*) + \rho]) < 0$, which implies in turn that $f(\bar{x}) < f(x_1^*)$, that is $\bar{x} \in S_2$. In conclusion the thesis follows by setting $\bar{\tau} = \max\{\tau', \tau^*\}$. □

4. Preliminary numerical results

Although the focus of our paper is more theoretical than computational, we have performed some tests to have an initial feeling of the practical interest of the filled functions proposed. We have restricted our attention to the filled function $V(x, \tau, \rho)$ in the case of box constrained minimization problems.

In particular we have set $\eta(t) = \gamma_1 t$ and $\varphi(t) = \gamma_2 \arctan(t)$, where γ_1 and γ_2 are scaling factors and hence we have used the following filled function:

$$V(x, \tau, \rho) = \gamma_1 \frac{1}{2} \|x - x_0\|^2 + \gamma_2 \arctan(\tau[f(x) - f(x_1^*) + \rho]) .$$

We have considered this filled function for its additive structure, which, as we have indicated in Section 2, should allow us to manage more easily the possible parameters of the filled function. The choice of testing on box constrained problems follows from the fact that for this class of problems we can relax the hypothesis on $\varphi(t)$, in particular we can replace the property (4) with the property (4b). This allows us to use the function $\varphi(t) = \gamma_2 \arctan(t)$, better conditioned than the ones which include exponential terms and less flat than $\varphi(t) = \min\{t, 0\}^3$. As regards our implementation we have set $\gamma_1 = 1$, $\gamma_2 = 10^6$, $\rho = 10^{-3}$ and $\tau = 10^4$. The point x_0 is initially chosen at random at the interior of the feasible set, and then it is updated whenever the local minimizaiton of the filled function gives a point x_τ where $f(x_\tau) \geqslant f(x_1^*)$ and $V(x_\tau, \tau, \rho) < V(x_0, \tau, \rho)$. In this case x_τ becomes the new x_0. We have stopped our algorithm whenever the local minimization of the filled function gives x_τ such that $\|x_\tau - x_0\| < 10^{-4}$ for $5n$ times running (where n is the dimension of the test problem). The rational behind this stopping criterion is the property that if the current local minimum x_1^* is the global minimum of $f(x)$, x_τ is the only global minimum of the filled function and it is very close to x_0.

Local minimizations have been performed by using the derivative free algorithm proposed in Lucidi and Sciandrone (1999) and the starting points of the minimizations have been generated at random in the feasible set.

We have used our code for solving 11 global optimization test problems, taken from literature (see, for example, Lucidi and Piccioni, 1989; and Brachetti et al., 1997)), with n ranging from 2 to 10. Each test problem has been solved 10 times with 10 different initial x_0. The obtained results are reported in Table 1, where

- Prob. is the name of the test function
- n is the dimension of the test function
- nf is the mean number of function evaluations needed to get the global minimum

Table 1. Numerical results obtained by using the filled function $V(x, \tau, \rho)$

Prob.	n	nf(nf*)	nm	nmF(nmF*)	f*	Fail
SHC	2	156.6(850.8)	1.2	0.2(10.2)	−1.03	0
5*n*	2	282.3(1101)	2.1	1.7(12.5)	0.54D − 15	0
5*n*	4	427(3903)	1.4	0.9(22.4)	0.11D − 13	0
5*n*	6	430.3(8347)	1.2	0.3(31.9)	0.12D − 13	0
5*n*	8	1093(15010)	1.7	1.3(45.2)	0.17D − 13	0
5*n*	10	973.1(22760)	1.3	0.9(55.6)	0.65D − 14	0
10*n*	2	141.7(966.5)	1.1	0.2(10.7)	0.11D − 11	0
10*n*	4	654.4(4045)	1.5	2.5(25.9)	0.17D − 12	0
10*n*	6	1249(9071)	1.8	2.7(36.8)	0.7D − 12	0
10*n*	8	1539(15440)	1.5	2.8(46.8)	0.41D − 12	0
10*n*	10	2191(24310)	1.5	3.3(61.7)	0.77D − 13	0
15*n*	2	257.6(1043)	1.8	1.1(11.2)	0.33D − 13	0
15*n*	4	411.4(3846)	1.4	0.9(21.7)	0.74D − 14	0
15*n*	6	1049(9084)	1.7	2.2(34.2)	0.1D − 12	0
15*n*	8	1021(15880)	1.6	1.5(44.4)	0.37D − 13	0
15*n*	10	2262(26160)	2	3.1(58.1)	0.16D − 14	0
Sh(5)	4	1189(4225)	2.4	7.8(31.89)	−10.15	1
Sh(7)	4	787.2(3815)	2.7	3.1(25.8)	− 10.4	0
Sh(10)	4	603.7(3706)	2.2	2(25.2)	−10.54	3
Hart	3	784.7(2323)	1.8	1.1(28.3)	−3.86	0
Hart	6	518.6(7797)	2.3	0.9(82.3)	−3,32	0
GandP	2	307.9(1225)	1.8	0.6(12.9)	3.00	0
CosMix	2	241.8(857.9)	1.6	1.5(11.5)	−0.2	0
CosMix	4	928.7(3255)	2.2	4.9(24.9)	−0.40	0
Qua	2	196.7(986.6)	1.5	0.6(10.8)	−0.35	0
Shu	2	345.8(1158)	2.111	2.6(14.6)	−186.7	1
Gri	2	890.4(1823)	3.5	7.3(17.3)	0.22D − 15	1

- nf* is the mean number of function evaluations needed to satisfy the stopping criterion
- nm is the mean number of local minimizaitons of the objective function $f(x)$
- nmF is the mean number of local minimizations of the filled function needed to get the global minimum
- nmF* is the mean number of local minimizations of the filled function needed to satisfy the stopping criterion
- f* is the obtained optimal function value
- fail. is the number of times where the stopping criterion is satisfied without having located the global minimum.

All the mean values have been computed without considering the failures.

Results reported seem to show the practibility of the approach, in fact the implemented algorithm has been able almost always to find the global minimum within an acceptable number of functions evaluation. Better results can be obtained by defining more sophisticated algorithm, which should include updating rules for the parameters τ, ρ and x_0, more efficient stopping criteria and tuning processes for the scaling factors.

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